A Simple, Adjustably Robust, Dynamic Portfolio Policy under Expected Return Ambiguity

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- A core subject of mathematical finance and operations research for six decades

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A core subject of mathematical finance and operations research for six decades

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Background on Portfolio Selection

- A core subject of mathematical finance and operations research for six decades
- starting with the ground-breaking work of Markowitz who initiated the Mean-Variance (MV) portfolio theory.
- An alternative approach takes the view that the investor is maximizing an expected utility function of his final wealth.
- I shall use the expected utility approach in the present paper.
Background on Multi-period Portfolio Management

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- The solution looks complicated.
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Connections to Robust Optimization

- I shall apply the Adjustable Robust Optimization (ARO) paradigm to the dynamic portfolio selection problem under mean return ambiguity.
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- Under the setting of the present paper I obtain a closed-form, partially myopic dynamic portfolio policy.
- The optimal policy is also affine in the estimate of the mean return.
The Setting

- There are $n$ risky assets with return vector $\mathbf{X}$. 

[Content continues on subsequent pages]
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- There are \( n \) risky assets with return vector \( \mathbf{X} \)
- which follows a Gaussian law with mean \( \mathbf{Y} \) and positive definite variance-covariance matrix \( \Sigma \).
- There is also a risk-less asset with return \( R \geq 1 \).
- We assume that the investor has a CARA utility, e.g. a negative exponential utility function.
The mean $\mathbf{Y}$ of the return vector $\mathbf{X}$ takes values in the ellipsoidal ambiguity set around the nominal (or, estimated) expected return vector $\bar{\mathbf{X}}$:

$$U_X = \{ \mathbf{Y} | \| \Sigma^{-1/2} (\mathbf{Y} - \bar{\mathbf{X}}) \|_2 \leq \sqrt{\epsilon} \},$$
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where $\epsilon$ is some positive number referred to as the ambiguity radius.
Ambiguity

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- It is well-known (DeMiguel and Nogales (2007)) that portfolio weights are very sensitive to imprecision in mean return.
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Protecting against ambiguity in the portfolio choice leads to more stable portfolios delivering a higher out-of-sample Sharpe ratio compared to classical Markowitz portfolios Garlappi et al. (2007).
The following experiment is based on DeMiguel/Nogales (2007).

Given are the (theoretical) moments of monthly returns

\[ \hat{\mu} = \begin{pmatrix} 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \end{pmatrix}, \quad \Sigma = 0.04 \times I_{N \times N}. \]

The Mean-Variance optimal weights for any risk aversion coefficient are 0.25 for each asset!\(^1\)

\(^1\)To verify this, set \( \hat{\mu} = a1 \) and \( \Sigma = bI \) in Eq. (3).
A simple experiment cont’d

Generate 140 returns $r_i \sim \mathcal{N}(\mu, \Sigma)$, $i = 1 \ldots 140$.

For different values of $t$ calculate sample moments from

$$\hat{\mu} = \frac{1}{120} \sum_{i=t}^{120+t-1} r_i, \quad \hat{\Sigma} = \frac{1}{120 - 1} \sum_{i=t}^{120+t-1} (r_i - \hat{\mu})(r_i - \hat{\mu})'$$

Rolling window estimation:

Rebalancing date $t = 1$: Calculate $\hat{\mu}$ and $\hat{\Sigma}$ using $r_1 \ldots r_{120}$
Rebalancing date $t = 2$: Calculate $\hat{\mu}$ and $\hat{\Sigma}$ using $r_{2} \ldots r_{121}$

...until $t = 20$.

In the following, for each $\hat{\mu}$, $\hat{\Sigma}$ optimal weights are calculated.
A simple experiment III

\[ \bar{\mu} \text{ and } \Sigma \text{ are unknown:} \]

\[ \bar{\mu} \text{ and } \Sigma \text{ are estimated using 120 monthly returns } r_i \text{ (rolling window).} \]

The dashed line shows optimal portfolio weights without estimation error \((= 0.25)\) and \(\gamma = 1.\)
A simple experiment IV

\[ \bar{\mu} \] is estimated and \[ \Sigma \] is known:

![Weights and rebalancing dates graph]

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The Single Period Problem

The Multiple-Period Problem

A simple experiment V

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\( \bar{\mu} \) is known and \( \Sigma \) is estimated:

![Graph showing weights over rebalancing dates with \( T=120 \)]
Statistical Motivation for Ellipsoidal Ambiguity

- The ellipsoidal uncertainty set is justified according to Stambaugh
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The quantity

\[ \frac{T(T-n)}{(T-1)n} (Y - \bar{X})^T \Sigma^{-1} (Y - \bar{X}) \]

has a \( \chi^2 \) distribution with \( n \) degrees of freedom, where \( T \) is the number of past observations for the asset prices.
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  has a $\chi^2$ distribution with $n$ degrees of freedom, where $T$ is number of past observations for the asset prices.
- The ambiguity restriction
  \[ \frac{T(T - n)}{(T - 1)n} (\mathbf{Y} - \mathbf{\bar{X}})^T \Sigma^{-1} (\mathbf{Y} - \mathbf{\bar{X}}) \leq \varepsilon \]
  can be interpreted as a “quantile” constraint for some quantile $\varepsilon$. 

PINAR
Robust Dynamic Portfolio Management under Ambiguity
The Single-Period Model

- The investor allocates a capital \( W_0 \) to be invested in the set of risky assets and the risk-less asset.
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$$W(\omega) = \omega^T X + [W_0 - 1^T \omega] R. \quad (1)$$
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- $\omega$ is $n$-dimensional vector representing the allocation in the risky assets and $1$ is a $n$-vector of ones.
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- $\omega$ is $n$-dimensional vector representing the allocation in the risky assets and $1$ is a $n$-vector of ones.
- The single period investor is interested in determining $\omega^*$ as an ambiguity-robust portfolio allocation in the sense that it solves the following problem
  \[
  \max_{\omega} \left\{ \min_{Y \in U_X} \mathbb{E}[-e^{-\alpha W(\omega)}] \right\} 
  \]

(2)

where $\alpha$ is a positive constant.
The Optimal Portfolio Choice

**Proposition**

A closed form solution for the problem (2) is obtained as

\[
\omega^* = \begin{cases} 
\left( \frac{\sqrt{H} - \sqrt{\epsilon}}{\alpha \sqrt{H}} \right) \Sigma^{-1} \mu & \text{if } H > \epsilon \\
0 & \text{o.w.}
\end{cases}
\]

where \( H = \mu^T \Sigma^{-1} \mu \) and \( \bar{\mu} = \bar{X} - R1. \)
Properties of the Optimal Portfolio Choice

- While the investor’s risk aversion or appetite is controlled by the parameter $\alpha$, the parameter $\epsilon$ controls the ambiguity aversion.
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- For $\epsilon = 0$ the ambiguity aversion is nil and we recover the optimal mean-variance portfolio $\omega_M$ that is also obtained by solving

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  The solution $\omega_M$ is known to be
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The solution $\omega_M$ is known to be

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- On the other hand, for fixed $\alpha$, when $\epsilon$ tends to $H$, we obtain a risk-less portfolio since all wealth is invested into the risk-less asset.
The ambiguity-robust portfolio (3) is mean-variance efficient.
Further Properties of the Optimal Portfolio Choice

- The ambiguity-robust portfolio $(3)$ is mean-variance efficient.
- Furthermore, the factor $\sqrt{H}$ is precisely the slope of the Capital Market Line (CML) in the MV portfolio theory.
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Partial separation has the consequence that optimal portfolios are affine or linear functions of the initial wealth $W_0$. 
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Furthermore, the factor $\sqrt{H}$ is precisely the slope of the Capital Market Line (CML) in the MV portfolio theory.

The ambiguity-robust portfolio is partially separated

Partial separation has the consequence that optimal portfolios are affine or linear functions of the initial wealth $W_0$.

In this case, the optimal portfolio rule is simply a constant as a function of initial wealth since it is completely independent of it.
For simplicity I develop initially the result for a two-period problem.
A Two-Period Optimal Portfolio Choice Problem

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- There are two time points $t = 0, 1$ at which the portfolio choice is made, the end of the time horizon $t = 2$ (in general $t = T$) is the moment where the final realized portfolio value, say $W_2$ ($W_T$) is revealed.
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- \( X_2 \) follows a Gaussian law with mean \( Y_2 \) and positive definite variance-covariance matrix \( \Sigma_2 \).
- For simplicity, the risk-less asset is assumed to have invariant per period return \( R \geq 1 \).
A Two-Period Optimal Portfolio Choice Problem under Ambiguity

I confine the mean $\mathbf{Y}_1$ of the first period return vector $\mathbf{X}_1$ to take values in the ellipsoidal ambiguity set around $\bar{\mathbf{X}}$:

$$U_{\bar{\mathbf{X}}}^1 = \{ \mathbf{Y}_1 | \| \Sigma_1^{-1/2} (\mathbf{Y}_1 - \bar{\mathbf{X}}_1) \|_2 \leq \sqrt{\epsilon} \},$$

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A Two-Period Optimal Portfolio Choice Problem under Ambiguity

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$$U^1_{\bar{\mathbf{X}}} = \{ \mathbf{Y}_1 | \| \Sigma_1^{-1/2} (\mathbf{Y}_1 - \bar{\mathbf{X}}_1) \|_2 \leq \sqrt{\epsilon} \}$$

and

- the mean $\mathbf{Y}_2$ of the second period return vector $\mathbf{X}_2$ to take values in the ellipsoidal ambiguity set around $\bar{\mathbf{X}}$:

$$U^2_{\bar{\mathbf{X}}} = \{ \mathbf{Y}_2 | \| \Sigma_2^{-1/2} (\mathbf{Y}_2 - \bar{\mathbf{X}}_2) \|_2 \leq \sqrt{\epsilon} \}.$$
A Two-Period ARO Portfolio Choice Problem

- The ARO dynamic portfolio choice

\[ V_2 = \max_{\omega_2} \min_{Y_2 \in U_2^2} \mathbb{E}_1[-e^{-\alpha W_2(\omega_2)}] \] (6)

Where \( W_2(\omega_2) = \omega_2 T + \left[W_1 - 1 \cdot \omega_2\right] R\), \( \mathbb{E}_1 \) denotes expectation after period 1 has elapsed, and

\[ V_1 = \max_{\omega_1} \min_{Y_1 \in U_1} \mathbb{E}_1[\mathbb{E}_2[V_2]] \], (7)

\( W_1 \) is the portfolio value at the end of period 1, the observer at time \( t = 1 \) will have already observed \( W_1 \), therefore, at the moment of making the choice for \( \omega_2 \) (i.e., beginning of stage 2) \( W_1 \) is no longer stochastic.
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Solving Backwards: Period 2

The following problem is solved:

$$\max_{\omega_2} -e^{-\alpha[\omega_2^T \tilde{X}_2 - R\omega_2^T 1 - \frac{\alpha}{2} \omega_2^T \Sigma_2 \omega_2 - \sqrt{\epsilon \omega_2^T \Sigma_2 \omega_2} + W_1 R]}.$$  
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By Proposition 1, the solution $\omega_2^*$ is given by

$$\omega_2^* = \left( \frac{\sqrt{H_2} - \sqrt{\epsilon}}{\alpha \sqrt{H_2}} \right) \Sigma^{-1} \bar{\mu}_2 \hspace{1cm} (9)$$

where $H_2 = \bar{\mu}_2^T \Sigma^{-1}_2 \bar{\mu}_2$ and $\bar{\mu}_2 = \tilde{X}_2 - R \mathbf{1}$, provided that $H_2 > \epsilon$. 
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This solution gives

$$V_2 = -e^{-\alpha[(\gamma - \frac{\alpha}{2} \gamma^2)H_2 - \sqrt{\epsilon H_2} + W_1 R]} ,$$

where $\gamma = \frac{\sqrt{\epsilon + H_2} - \sqrt{\epsilon}}{\alpha \sqrt{\epsilon + H_2}}$. 

PINAR Robust Dynamic Portfolio Management under Ambiguity
Solving Backwards: Period 1

The problem of computing $V_1$

$$V_1 = \max_{\omega_1} \min_{Y_1 \in U^1_{\bar{X}}} \mathbb{E} \left[ e^{-\alpha \left( \gamma - \frac{\alpha^2}{2} \right) H_2 - \sqrt{\epsilon} H_2 + W_1 R } \right]$$

where $W_1 = \omega_1^T X_1 + [W_0 - 1^T \omega_1] R$. 

One can easily continue the backward process when $T > 2$. 

Solving Backwards: Period 1

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  where $W_1 = \omega_1^T X_1 + [W_0 - 1^T \omega_1] R$.

- Hence solve
  \[ \max_{\omega_1} -e^{-\alpha \kappa} e^{-\alpha R \left[\omega_1^T \bar{X}_1 - R \omega_1^T 1 - \frac{\alpha R}{2} \omega_1^T \Sigma_1 \omega_1 - \sqrt{\epsilon \omega_1^T \Sigma_1 \omega_1} + W_0 R\right]} \]
  where I defined $\kappa = (\gamma - \frac{\alpha}{2} \gamma^2)H_2 - \sqrt{\epsilon}H_2$. 

One can easily continue the backward process when $T > 2$. 

PINAR Robust Dynamic Portfolio Management under Ambiguity
Solving Backwards: Period 1

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\[
V_1 = \max_{\omega} \min_{Y_1 \in U_1^1} \mathbb{E}[e^{-\alpha[(\gamma - \frac{\alpha}{2}\gamma^2)H_2 - \sqrt{\epsilon H_2} + W_1 R]}]
\]

where $W_1 = \omega_1^T X_1 + [W_0 - 1^T \omega_1] R$.

- Hence solve

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\max_{\omega_1} -e^{-\alpha \kappa} e^{-\alpha R[\omega_1^T \bar{X}_1 - R\omega_1^T 1 - \frac{\alpha R}{2} \omega_1^T \Sigma_1 \omega_1 - \sqrt{\epsilon \omega_1^T \Sigma_1 \omega_1} + W_0 R]}
\]

where I defined $\kappa = (\gamma - \frac{\alpha}{2}\gamma^2)H_2 - \sqrt{\epsilon H_2}$.

- The optimal portfolio $\omega_1^*$ for $H_1 > \epsilon$

\[
\omega_1^* = \left( \frac{\sqrt{H_1} - \sqrt{\epsilon}}{\alpha R \sqrt{H_1}} \right) \Sigma_1^{-1} \bar{\mu}_1.
\]
Solving Backwards: Period 1

- The problem of computing $V_1$

$$V_1 = \max_{\omega_1} \min_{Y_1 \in U\bar{X}_1} \mathbb{E}[e^{-\alpha[(\gamma - \frac{\alpha}{2} \gamma^2)H_2 - \sqrt{\epsilon H_2} + W_1 R]}]$$

where $W_1 = \omega_1^T X_1 + [W_0 - \mathbf{1}^T \omega_1] R$.

- Hence solve

$$\max_{\omega_1} -e^{-\alpha \kappa} e^{-\alpha R[\omega_1^T \bar{X}_1 - R\omega_1^T \mathbf{1} - \frac{\alpha R}{2} \omega_1^T \Sigma_1 \omega_1 - \sqrt{\epsilon \omega_1^T \Sigma_1 \omega_1 + W_0 R}]}$$

where I defined $\kappa = (\gamma - \frac{\alpha}{2} \gamma^2)H_2 - \sqrt{\epsilon H_2}$.

- The optimal portfolio $\omega_1^*$ for $H_1 > \epsilon$:

$$\omega_1^* = \left( \frac{\sqrt{H_1} - \sqrt{\epsilon}}{\alpha R \sqrt{H_1}} \right) \Sigma_1^{-1} \bar{\mu}_1.$$ \hspace{1cm} (10)

- One can easily continue the backward process when $T > 2$. 
I choose a distinct ambiguity radius $\epsilon_t$ for each period $t = 1, \ldots, T$

\[
\max_{\omega_t} -K_t e^{-\alpha \prod_{j=t+1}^T R_j [\omega_t^T \bar{X}_t - R_t \omega_t^T 1] - \frac{\alpha \prod_{j=t+1}^T R_j}{2} \omega_t^T \Sigma_t \omega_t - \sqrt{\epsilon_t \omega_t^T \Sigma_t \omega_t + W_{t-1} R_t}}
\]

for some positive constant $K_t$.  

(11)
A More General Case

- I choose a distinct ambiguity radius $\epsilon_t$ for each period $t = 1, \ldots, T$.
- Assume a varying risk-less rate (known a priori) $R_t$ for each period $t = 1, \ldots, T$.

The optimal portfolio choice $\omega_t^*$ at stage $t$ is obtained by solving

$$
\max_{\omega_t} -K_t e^{-\alpha \prod_{j=t+1}^T R_j [\omega_t^T \bar{X}_t - R_t \omega_1^T 1 - \frac{\alpha \prod_{j=t+1}^T R_j}{2} \omega_t^T \Sigma_t \omega_t - \sqrt{\epsilon_t \omega_t^T \Sigma_t \omega_t} + W_{t-1} R_t]}
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(11)
The ARO Problem in $T$ Periods

The adjustable robust dynamic portfolio problem is

$$V_T = \max_{\omega_T} \min_{Y_T \in U_X^T} \mathbb{E}_{T-1}[-e^{-\alpha W_T(\omega_T)}]$$  \hspace{1cm} (12)$$

$$V_{T-1} = \max_{\omega_{T-1}} \min_{Y_{T-1} \in U_X^{T-1}} \mathbb{E}_{T-2}[V_{T-1}],$$  \hspace{1cm} (13)$$

$$\vdots$$

$$V_1 = \max_{\omega_1} \min_{Y_1 \in U_X^1} \mathbb{E}[V_2],$$  \hspace{1cm} (14)$$
The adjustable robust dynamic portfolio policy is given by

$$\omega^*_t = \frac{\sqrt{H_t} - \sqrt{\epsilon_t}}{\alpha(\prod_{j=t+1}^{T} R_j)\sqrt{H_t}} \Sigma_t^{-1} \bar{\mu}_t, \ t = 1, \ldots, T, \quad (15)$$

where $\bar{\mu}_t = \tilde{X}_t - R_t 1$, provided that $\epsilon_t < H_t$ for all $t = 1, \ldots, T$. 
Properties of the Optimal Policy

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- In fact, the optimal policy at period $t$ is the same as the one that would be used if the investor were dealing with a single period problem where he chooses $\omega_t$ to maximize the robust expected wealth at the end of $t$ and subsequently re-invest the realized wealth, say $\tilde{W}_t$, at the rates $R_{t+1}, R_{t+2}, \ldots, R_T$, exclusively in the risk-less asset.
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- Such a policy is called a partially myopic policy; Mossin (1968).
I.e., the investor solves

\[
\max_{\omega_t} \min_{Y_t \in U^t_X} \mathbb{E}[-K_t e^{-\alpha(\prod_{j=t+1}^T R_j)} W(\omega_t)]
\]
Properties of the Optimal Policy (cont’d)

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- where \( W(\omega_t) = \omega_t^T X_t + [W_{t-1} - 1^T \omega_t] R_t \),
Properties of the Optimal Policy (cont’d)

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- where \( W(\omega_t) = \omega_t^T X_t + [W_{t-1} - 1^T \omega_t] R_t \),

- and the result is exactly the maximization problem in (11).
Properties of the Optimal Policy: Special Case

- The portfolio strategy obtained by substituting $\epsilon = 0$

$$\omega^*_t = \frac{1}{\alpha(\prod_{j=t+1}^{T} R_j)} \Sigma_t^{-1} \bar{\mu}_t, \quad t = 1, \ldots, T,$$
The portfolio strategy obtained by substituting $\epsilon = 0$

$$\omega_t^* = \frac{1}{\alpha(\prod_{j=t+1}^T R_j)} \sum_{t}^{-1} \mu_t, \ t = 1, \ldots, T,$$

is precisely the multi-period optimal strategy of Mossin (1968)
The optimal portfolio policy is an *affine* function of the period $t$ nominal expected return $\bar{X}_t$:

$$\omega^*_t = \frac{\sqrt{H_t} - \sqrt{\epsilon_t}}{\alpha(\prod_{j=t+1}^{T} R_j)\sqrt{H_t}} \Sigma_t^{-1}(\bar{X}_t - R_t1), \ t = 1, \ldots, T. \quad (16)$$
Another View of the Optimal Policy: Affine Policy

- The optimal portfolio policy is an *affine* function of the period $t$ nominal expected return $\bar{X}_t$:

$$\omega_t^* = \frac{\sqrt{H_t} - \sqrt{\epsilon_t}}{\alpha(\prod_{j=t+1}^{T} R_j)^{\sqrt{H_t}}} \sum_t^{-1}(\bar{X}_t - R_t \mathbf{1}), \ t = 1, \ldots, T. \quad (16)$$

- Thus, we can assert that the optimal portfolio policy is an *affinely adjustable* robust portfolio policy,
The optimal portfolio policy is an *affine* function of the period $t$ nominal expected return $\bar{X}_t$:

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Another View of the Optimal Policy: Affine Policy

- The optimal portfolio policy is an *affine* function of the period *t* nominal expected return $\tilde{X}_t$:

$$
\omega_t^* = \frac{\sqrt{H_t} - \sqrt{\epsilon_t}}{\alpha(\prod_{j=t+1}^{T} R_j)\sqrt{H_t}} \Sigma_t^{-1}(\tilde{X}_t - R_t1), \ t = 1, \ldots, T. \quad (16)
$$

- Thus, we can assert that the optimal portfolio policy is an *affinely adjustable* robust portfolio policy,

- Usually, the procedure is reversed.

- One formulates an ARO problem and approximates it using an affine policy.